This is an example of an NSF GRFP Research Statement from a student who was awarded the fellowship as a senior undergraduate in 2013.

On the Theory of Poset Games

Briefly introduce the topic of research and the problem to solve and its impact

Mathematics, like nearly all knowledge-based studies, progresses by enhancement and extension, rather than through bold Eureka moments. Problems that have fascinated society for centuries, such as Fermat's Last Theorem, were solved by building a library of tools and proofs and integrating them to solve a large, seemingly intractable problem. Determining the best strategy for tic-tac-toe is trivial because there are so few combinations, but combinatorial games such as chess, checkers, and Go have long captivated human imagination due to their intrinsic difficulty. Furthermore, the problems that arise in combinatorial game theory are mathematically non-trivial and can have significant ramifications for technological progress in our society. For instance, game theoretic techniques have been used to create efficient error correcting codes for data transmission [2].

Provide relevant background, the gap and the specific aims with proposed method & technique – be specific!

A poset game is a specific type of two-player combinatorial game in which players alternate choosing an element of a partially ordered set, or poset. Each chosen element is removed along with all elements greater than it. The first player unable to select an element of the poset loses. One of the intriguing properties of poset games is that deciding who should win a poset game is often much easier than actually giving the winning sequence of moves for that player. My research is concerned with understanding the nature of these types of discrepancies. Although these fundamental questions about poset games are interesting in their own right, the techniques and ideas required to solve these problems will have a much broader impact. For example, due to the natural aspect of competition in poset games, this research could yield insight into economic models in which consumers attempt to maximize their utility through some finite action space.

Poset games have been studied in various forms since a complete analysis of the game of Nim was given in 1901 by C. Bouton. However, until recently, relatively little was known about the complexity of an arbitrary poset game. In [1], Deuber and Thomassé give a polynomial time algorithm for finding the winner of *N*-free posets, and in [5], Kalinich shows that computing the winner of an arbitrary poset game is at least as hard as NC^1 under AC^0 reductions. However, these results still left the complexity of arbitrary posets games quite open. In [3], I prove that poset games are PSPACE-complete. This definitively solves the problem of arbitrary poset game complexity and shows that deciding the winner of an arbitrary poset game is almost assuredly computationally difficult.

The proof of the theorem uses the PSPACE-complete game Node Kayles, in which players vie to choose an independent set of a graph [6]. I give a reduction that translates an instance of Node Kayles into a poset game such that Player 1 has a winning strategy for the poset game if and only if Player 1 has a winning strategy for the corresponding Nodes Kales game. Because poset games can be solved in PSPACE by a straightforward enumeration of all possible game trees, the above construction shows that they are PSPACE-complete.

The construction in my proof yields a poset game with three levels. That is, there are no $x_1,x_2,\ldots,x_n$ such that $x_1 < x_2 < \ldots < x_n$ for $n > 3$. the complexity of two-level posets. Thus, we can conclude that poset games with three or more levels are difficult to solve. In contrast, one-level posets can be solved trivially by finding the parity of the elements. However, the complexity of two-level poset games remains unknown. Thus, one of my goals for this research is to determine definitively I first plan to look for patterns in two-level poset games by investigating poset games with few bottom elements, that is, those elements x such that y 6= x implies y 6≤ x . Recent work by Stephen Fenner, Rohit Gurjar, Arpita Korwar, and Thomas Thierauf shows that it is enough to consider only finitely many two-level posets for a given number of bottom elements. Thus, for small numbers of bottom elements, we can look for patterns by enumerating the *g*-numbers of all possible two-level games. The *g*-number is an important property of many two-player games that not only indicates which player shall win, but also gives a slight indication of the underlying structure of the game.

This technique has already been applied for very small numbers of bottom points, but enumerating all possible poset games becomes cumbersome even when only considering four bottom elements. Dr. Fenner and I have noticed that some of the upper elements in the poset are seemingly more important than other upper elements of the poset in that their presence or absence has a noticeably larger impact on the *g*-number of the game. One such element, the upper element above all other bottom elements, acts like a pass for one of the players. That is, choosing it is equivalent to passing one's turn. This, of course, is not allowed under normal play. However, games with passes are becoming an increasingly large subdomain within combinatorial game theory. Thus, I hope use techniques such as those described in [4] to help determine when the presence of a pass allows the first player to win. This is a novel approach to solving two-level poset games that could yield insight into areas where more traditional approaches have been insufficient. After developing these techniques, I plan to apply them to other important problems in combinatorial game theory.

State you interests for the project, clearly describe your research problem, specific aims (be technical) and expected research outcomes to demonstrate your knowledge in the field.

I am also interested in classifying posets played over well-defined structures such as the boolean lattice. A longstanding open problem restated in [1] is as follows: is the g-number of a boolean lattice without its minimum and maximum elements equal to 0? I found an explicit algorithm to verify this conjecture for fewer than six dimensions, and wrote a program along with Dr. Fenner to verify the conjecture in the six-dimensional case. That said, Dr. Fenner and I now hope to gain insight into this problem by first looking at two-level subposets of the boolean lattice. We have already found solutions for many subposets by exploiting a certain recursive structure that they possess. We believe that by building a library of known two-level subposets we can better construct solutions in the general case. Since many popular combinatorial games, such as Chomp, can be viewed as games played over partially ordered sets, an analysis of poset games is a very important area in combinatorial game theory. Furthermore, by drawing connections between games played over two-level posets and games with a pass, this research has the potential to yield an even broader understanding of classical game theoretic techniques.

References

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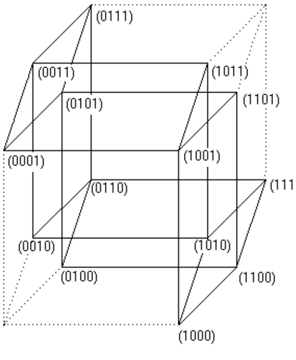

Figure 1: Boolean lattice without its minimum and maximum elements.